CHAPTER 7: OPTIMAL RISKY PORTFOLIOS

Solutions to Suggested Problems

1. (a) and (e). Short-term rates and labor issues are factors that are common to all firms and therefore must be considered as market risk factors. The remaining three factors are unique to this corporation and are not a part of market risk.

2. (a) and (c). After real estate is added to the portfolio, there are four asset classes in the portfolio: stocks, bonds, cash, and real estate. Portfolio variance now includes a variance term for real estate returns and a covariance term for real estate returns with returns for each of the other three asset classes. Therefore, portfolio risk is affected by the variance (or standard deviation) of real estate returns and the correlation between real estate returns and returns for each of the other asset classes. (Note that the correlation between real estate returns and returns for cash is most likely zero.)

4. The parameters of the opportunity set are:

\[ E(r_S) = 20\%, \quad E(r_B) = 12\%, \quad \sigma_S = 30\%, \quad \sigma_B = 15\%, \quad \rho = 0.10 \]

From the standard deviations and the correlation coefficient we generate the covariance [note that \( \text{Cov}(r_S, r_B) = \rho \times \sigma_S \times \sigma_B \):]

\[ \text{Cov}(r_S, r_B) = 45 \]

The minimum-variance portfolio is computed as follows:

\[
\begin{align*}
    w_{\text{Min}}(S) &= \frac{\sigma_B^2 - \text{Cov}(r_S, r_B)}{\sigma_S^2 + \sigma_B^2 - 2 \text{Cov}(r_S, r_B)} = \frac{225 - 45}{900 + 225 - (2 \times 45)} = 0.1739 \\
    w_{\text{Min}}(B) &= 1 - 0.1739 = 0.8261
\end{align*}
\]

The minimum variance portfolio mean and standard deviation are:

\[
\begin{align*}
    E(r_{\text{Min}}) &= (0.1739 \times 20\%) + (0.8261 \times 12\%) = 13.39\% \\
    \sigma_{\text{Min}} &= \left[ w_S^2 \sigma_S^2 + w_B^2 \sigma_B^2 + 2w_S w_B \text{Cov}(r_S, r_B) \right]^{1/2} \\
    &= \left[ (0.1739^2 \times 900) + (0.8261^2 \times 225) + (2 \times 0.1739 \times 0.8261 \times 45) \right]^{1/2} \\
    &= 13.92\%
\end{align*}
\]
5. | Proportion in Stock Fund (%) | Proportion in Bond Fund (%) | Expected Return (%) | Standard Deviation (%) | 
--- | --- | --- | --- | --- | 
0.00 | 100.00 | 12 | 15 | 
17.39 | 82.61 | 13.3912 | 13.91746 | Min. var. portfolio | 
20.00 | 80.00 | 13.6 | 13.94274 | 
40.00 | 60.00 | 15.2 | 15.7035 | 
45.6423 | 54.3577 | 15.65138 | 16.6223 | Tangency portfolio | 
60.00 | 40.00 | 16.8 | 19.53458 | 
80.00 | 20.00 | 18.4 | 24.48265 | 
100.00 | 0.00 | 20 | 30 | 

(Note that I have added two extra portfolios in the table above to set apart the minimum variance portfolio and the tangency portfolio. I used Excel to calculate the exact value. In an exam environment only approximate values would suffice.)

Graph shown below.

6. The above graph indicates that the optimal portfolio is the tangency portfolio with expected return approximately 15.6% and standard deviation approximately 16.5%. (Note that this problem is for your understanding. If such question appears in the exam, then it would be designed such that the answer would be more discernible.)
7. The proportion of the optimal risky portfolio invested in the stock fund is given by:

\[ w_s = \frac{[E(r_s) - r_f] \times \sigma_B^2 - [E(r_f) - r_f] \times Cov(r_s, r_B)}{[E(r_s) - r_f] \times \sigma_B^2 + [E(r_f) - r_f] \times \sigma_S^2 - [E(r_s) - r_f + E(r_f) - r_f] \times Cov(r_s, r_B)} \]

\[ = \frac{[(20 - 8) \times 225] - [(12 - 8) \times 45]}{[(20 - 8) \times 225] + [(12 - 8) \times 900] - [(20 - 8 + 12 - 8) \times 45]} \]

\[ = 0.4516 \]

\[ w_B = 1 - 0.4516 = 0.5484 \]

The mean and standard deviation of the optimal risky portfolio are:

\[ E(r_p) = (0.4516 \times 20\%) + (0.5484 \times 12) = 15.61\% \]

\[ \sigma_p = \sqrt{[(0.4516^2 \times 900) + (0.5484^2 \times 225) + (2 \times 0.4516 \times 0.5484 \times 45)]} = 16.54\% \]

8. The reward-to-volatility ratio of the optimal CAL is:

\[ \frac{E(r_p) - r_f}{\sigma_p} = \frac{15.61 - 8}{16.54} = 0.46 \]

9. a. If you require that your portfolio yield an expected return of 14%, then you can find the corresponding standard deviation from the optimal CAL. The equation for this CAL is:

\[ E(r_c) = r_f + \frac{E(r_p) - r_f}{\sigma_p} \times \sigma = 8 + 0.46 \times \sigma \]

\[ \to \sigma_c = (14 - 8) / 0.46 = 13.04\% \]

If \( E(r_c) \) is equal to 14%, then the standard deviation of the portfolio is 13.04%.

b. To find the proportion invested in the T-bill fund, remember that the mean of the complete portfolio (i.e., 14%) is an average of the T-bill rate and the optimal combination of stocks and bonds (P). Let \( y \) be the proportion invested in the portfolio \( P \). The mean of any portfolio along the optimal CAL is:

\[ E(r_c) = (1 - y) \times r_f + y \times E(r_p) = r_f + y \times [E(r_p) - r_f] = 8 + y \times (15.61 - 8) \]

Setting \( E(r_c) = 14\% \) we find: \( y = 0.7884 \) and \( (1 - y) = 0.2116 \) (the proportion invested in the T-bill fund).
To find the proportions invested in each of the funds, multiply 0.7884 times the respective proportions of stocks and bonds in the optimal risky portfolio:

Proportion of stocks in complete portfolio = 0.7884 \times 0.4516 = 0.3560

Proportion of bonds in complete portfolio = 0.7884 \times 0.5484 = 0.4324

10. Using only the stock and bond funds to achieve a portfolio expected return of 14%, we must find the appropriate proportion in the stock fund ($w_S$) and the appropriate proportion in the bond fund ($w_B = 1 - w_S$) as follows:

$$14\% = 20\% \times w_S + 12\% \times (1 - w_S) = 12\% + 8\% \times w_S \Rightarrow w_S = 0.25$$

So the proportions are 25% invested in the stock fund and 75% in the bond fund. The standard deviation of this portfolio will be:

$$\sigma_P = [(0.25^2 \times 900) + (0.75^2 \times 225) + (2 \times 0.25 \times 0.75 \times 45)]^{1/2} = 14.13\%$$

This is considerably greater than the standard deviation of 13.04% achieved using T-bills and the optimal portfolio.